Tikrit university
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Lectures on
Numerical Analysis

Chapter 2
Solving a system of Linear Equations

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Linear Algebraic Equations

A system of equations consists of two or more equations with two or more variables, where any solution must satisfy all of the equations in the system at the same time.

The general form of a system of n linear algebraic equations is:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Matrix

What is a matrix?

•It is an array of elements that are arranged in orderly rows and columns.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

2^{nd} row

m^{th} column

Elements are indicated by a_{ij} (i: row, j: column)

Row vector:

$$[R] = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}_{1 \times n}$$

Column vector:

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{m \times 1}$$

You know what is meant by

- **Square matrix**

  [A]nxm is a square matrix if n=m.

- **Diagonal matrix**

  [A]nxn is diagonal if aij = 0 for all i=1,...,n ; j=1,...,n and i≠j

- **Identity matrix**

  [A]nxn is an identity matrix if it is diagonal with aii=1 i=1,...,n . Shown as [I]

- **Triangular matrix:**

  1. Upper triangular matrix: $[A]_{nxn}$ is upper triangular if $a_{ij}=0$ i=1,...,n ; j=1,...,n and i>j

  2. Lower triangular matrix: $[A]_{nxn}$ is lower triangular if $a_{ij}=0$ i=1,...,n ; j=1,...,n and i<j

Special Types of Square Matrices

$$[A]=\begin{bmatrix} 5 & 1 & 2 & 16 \\ 1 & 3 & 7 & 39 \\ 2 & 7 & 9 & 6 \\ 16 & 39 & 6 & 88 \end{bmatrix}$$

Symmetric

$$[D]=\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \vdots & \\ & & & a_{nn} \end{bmatrix}$$

Diagonal

$$[I]=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Identity

$$[A]=\begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ \vdots & & \ddots \\ a_{n1} & \cdots\cdots & a_{nn} \end{bmatrix}$$

Lower Triangular

$$[A]=\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ & a_{22} \cdots & a_{2n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}$$

Upper Triangular

Solving Small Numbers of Equations

Consider a linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

There are many ways to solve a system of linear equations:

1. Graphical method 2. Cramer's rule 3. Method of elimination

4. Numerical methods for solving larger number of linear equations

There are two numerical approaches for solving system of linear equations :

1. Direct elimination methods 2. Iterative methods

1. Direct elimination methods

As the name suggests the methods are having procedures of algebraic elimination of the contents in the coefficient matrix that lead to solution.

A) Gauss elimination B) Gauss-Jordan C) Matrix inverse D) LU factorization etc.

To perform elimination methods to find the solution of linear algebraic system we need to do row operations.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ \vdots \\ b_n \end{Bmatrix}$$

i.e., $A \quad x = b$

Solve $Ax = b$

Gaussian elimination

In mathematics, **Gaussian elimination**, also known as row reduction, is a method for **solving systems** of **linear equations**

Two steps	**1. Forward Elimination**	**2. Back Substitution**

1. Forward Elimination

reduces $Ax = b$ to an upper triangular system $Tx = b$

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Forward Elimination

2. *Back substitution* can then solve $Tx = b$ for x

$$\Downarrow$$

$$x_3 = \frac{b''_3}{a''_{33}} \qquad x_2 = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$$

$$x_1 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$$

Back Substitution

a. Forward Elimination

To solve the system of three equations in form :

$$E_1 : a_{11}x_1 \quad a_{12}x_2 \quad a_{1n}x_n = b_1 \qquad .... (1)$$
$$E_2 : a_{21}x_1 \quad a_{22}x_2 \quad a_{23}x_3 = b_2 \qquad .... (2)$$
$$E_3 : a_{31}x_1 \quad a_{32}x_2 . \quad a_{23}x_3 = b_3 \qquad .... (3)$$

Transform to an Upper Triangular Matrix

First step is eliminate x_1 from all equations(2) and (3), through multiply equation(1) by r_1 calculated as

$$r_1 = \frac{coeff.of\ x_1\ in\ eq.(2)}{coeff.of\ x_1\ in\ eq.(1)} = \frac{a_{21}}{a_{11}}$$

and subtracted it from eq.(2) as ; $eq.(2) - r_1\, eq.(1)$

$$(a_{21} - r_1 a_{11})x_1 + (a_{22} - r_1 a_{12})x_2 + (a_{23} - r_1 a_{13})x_3 = b_2 - r_1 b_1$$

But $a_{21} - \frac{a_{21}}{a_{11}} a_{11} = 0$

we get; $E_2' : a_{22}' x_2 + a_{23}' x_3 = b_2'$ .... (4)

as;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3 \end{bmatrix}$$

► In similar way x_1 is eliminate from eq.(3) by multiplying eq.(1) by r_2

$$r_2 = \frac{coeff.of\ x_1\ in\ eq.(3)}{coeff.of\ x_1\ in\ eq.(1)} = \frac{a_{31}}{a_{11}}$$

and subtracted it from eq.(3) as ; $eq.(3) - r_2\, eq.(1)$

$$(a_{31} - r_2 a_{11})x_1 + (a_{32} - r_2 a_{12})x_2 + (a_{33} - r_2 a_{13})x_3 = b_3 - r_2 b_1$$

also $$a_{31} - \frac{a_{31}}{a_{11}} a_{11} = 0$$

we get; $$E_3' : a_{32}' x_2 + a_{33}' x_3 = b_3' \qquad \ldots. (5)$$

and so $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Then eliminate x_2 from eq.(5) by multiply by r_3

$$r_3 = \frac{coeff.of\ x_2\ in\ eq.(5)}{coeff.of\ x_2\ in\ eq.(4)} = \frac{a_{32}'}{a_{22}'}$$

then subtracted it from eq.(5) as ; $eq.(5) - r_3\, eq.(4)$

$$(a_{32}' - r_3 a_{22}')x_2 + (a_{33}' - r_3 a_{23}')x_3 = b_3' - r_2 b_2'$$

and $$a_{32}' - \frac{a_{32}'}{a_{22}'} a_{22}' = 0$$

so $$E_3'' : a_{33}'' x_3 = b_3'' \qquad \ldots. (6)$$

and be ; $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3'' \end{bmatrix}$$

Now evaluate x_3 from eq.(6) as; $$x_3 = \frac{b_3''}{a_{33}''}$$

The backward substitution for value of x_3 in eq.(4) to evaluate x_2. Further substituting values of x_3 and x_3 in eq.(1) to evaluate x_1

Example: Solve the following equations system, using Gauss elimination method

$$2x_1 + 3x_2 + x_3 = 7$$
$$3x_1 + 4x_2 + 2x_3 = 11$$
$$4x_1 + x_2 + x_3 = 11$$

Solution: a) Forward Elimination

$$2x_1 + 3x_2 + x_3 = 7 \quad ....(1)$$
$$3x_1 + 4x_2 + 2x_3 = 11 \quad ....(2)$$
$$4x_1 + x_2 + 3x_3 = 11 \quad ....(3)$$

so will be $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 11 \end{bmatrix}$

$R_2 - r_1 R_1$

where $r_1 = \frac{a_{21}}{a_{11}} = \frac{3}{2}$

step 1

so ; $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -0.5 & 0.5 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0.5 \\ 11 \end{bmatrix}$ $R_3 - r_2 R_1$

where $r_2 = \frac{a_{31}}{a_{11}} = \frac{4}{2} = 2$

step 2

so ; $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -0.5 & 0.5 \\ 0 & -5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0.5 \\ -3 \end{bmatrix}$ $R_3 - r_3 R_2$

where $r_3 = \frac{a'_{32}}{a'_{22}} = \frac{-5}{-0.5}$

step 3

so ; $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -0.5 & 0.5 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0.5 \\ -8 \end{bmatrix}$ eq.(4)

b) Back Substitution

From matrix $x_3 = \frac{-8}{-4} = 2$,

put value of x_3 in eq.(4) $x_2 = 1$ and then substituting the values in eq.(1) $x_1 = 1$

Example: Solve the following equations system, using Gauss elimination method

$$\begin{aligned} 2x_1 + x_2 - x_3 + 2x_4 &= 5 \\ 4x_1 + 5x_2 - 3x_3 + 6x_4 &= 9 \\ -2x_1 + 5x_2 - 2x_3 + 6x_4 &= 4 \\ 4x_1 + 11x_2 - 4x_3 + 8x_4 &= 2 \end{aligned}$$

Solution:

a) Forward Elimination

so ;

$$\begin{aligned} 2x_1 + x_2 - x_3 + 2x_4 &= 5 \\ 4x_1 + 5x_2 - 3x_3 + 6x_4 &= 9 \\ -2x_1 + 5x_2 - 2x_3 + 6x_4 &= 4 \\ 4x_1 + 11x_2 - 4x_3 + 8x_4 &= 2 \end{aligned}$$

so will be

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 4 & 5 & -3 & 6 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 4 \\ 2 \end{bmatrix}$$

To eliminate x_1 from equations 2, 3, and 4,

so ;

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 4 & 5 & -3 & 6 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 4 \\ 2 \end{bmatrix}$$

$R_2 - r_1 \, R_1$ step 1

$R_3 - r_2 \, R_1$ step 2

$R_4 - r_3 \, R_1$ step 3

where $r_1 = \frac{a_{21}}{a_{11}} = \frac{4}{2}$

where $r_2 = \frac{a_{31}}{a_{11}} = \frac{-2}{2} = -1$

where $r_3 = \frac{a_{41}}{a_{11}} = \frac{4}{2}$

To eliminate x_2 from equations 3, and 4,

so ;
$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 6 & -3 & 8 \\ 0 & 9 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \\ -8 \end{bmatrix}$$

$R_3 - r_4 R_2$ step 5

$R_4 - r_5 R_2$ step 6

where $r_4 = \frac{a'_{32}}{a'_{22}} = \frac{6}{3}$

where $r_5 = \frac{a'_{42}}{a'_{22}} = \frac{9}{3}$

To eliminate x_3 from equations 4,

so ;
$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 11 \\ -5 \end{bmatrix}$$

$R_4 - r_6 R_3$ step 7

where $r_6 = \frac{a'_{43}}{a'_{33}} = \frac{1}{-1}$

so ;
$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 11 \\ 6 \end{bmatrix}$$

b) Back Substitution Solving by back substitution, we obtain

From matrix $x_4 = \frac{6}{2} = 3$, **So** $x_3 = 1$ $x_2 = -2$ and $x_1 = 1$

Gauss - Jordan method

Gauss-Jordan elimination is another method for solving systems of equations in matrix form. The **Gauss-Jordan Method** is similar to **Gaussian Elimination**, except that the entries both above and below each pivot are targeted (zeroed out).

Solution of Linear Algebraic Equations using the Gauss-Jordan Method requires a number of general steps

- Within each pass two steps are performed:

1. A normalization step to reduce a diagonal element to One.
2. An elimination step to reduce off diagonal elements in the same column as the normalized diagonal element to zeros in the other rows.

These steps can be achieved by

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

To understand the Gauss, we will work with a numerical example

Consider the following system of linear equations,

$$
\begin{aligned}
x_1 + x_2 + 2x_3 - 5x_4 &= 3 \\
2x_1 + 5x_2 - x_3 - 9x_4 &= -3 \\
2x_1 + x_2 - x_3 + 3x_4 &= -11 \\
x_1 - 3x_2 + 2x_3 + 7x_4 &= -5
\end{aligned}
$$

the coefficient matrix is

$$
\begin{bmatrix} 1 & 1 & 2 & -5 \\ 2 & 5 & -1 & -9 \\ 2 & 1 & -1 & 3 \\ 1 & -3 & 2 & 7 \end{bmatrix}
$$

The steps of the Gauss-Jordan method are :

Step 1:

Form the augmented matrix containing only the constants and right hand side of the equations

$$\begin{aligned} 3x_1 + 4x_2 + 6x_3 &= 9 \\ 7x_1 + 9x_2 + 10x_3 &= 2 \\ 1x_1 + 2x_2 + 5x_3 &= 7 \end{aligned}$$

$$\downarrow$$

$$\begin{bmatrix} 3 & 4 & 6 & 9 \\ 7 & 9 & 10 & 2 \\ 1 & 2 & 5 & 7 \end{bmatrix}$$

The Gauss-Jordan method uses the two properties mentioned earlier to reduce the augmented matrix

$$\begin{bmatrix} 3 & 4 & 6 & 9 \\ 7 & 9 & 10 & 2 \\ 1 & 2 & 5 & 7 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 1 & a_{34} \end{bmatrix}$$

- Solution:

$$x_1 = a_{14}$$

$$x_2 = a_{24}$$

$$x_3 = a_{34}$$

Gauss-Jordan Elimination- Example

Example : Solve using Gauss-Jordan elimination , the following equations system

$$2x_2 + x_4 = 0$$
$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$
$$4x_1 - 3x_2 + x_4 = -7$$
$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

Solution:-**a) Forward Elimination**

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array}\right] \begin{array}{c} R1 \rightleftarrows R4 \\ R4/6.0 \end{array} \left[\begin{array}{cccc|c} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R2 - 2 \cdot R1 \\ R3 - 4 \cdot R1 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 0 & 1.6667 & 5 & 3.6667 & -2 \\ 0 & -3.6667 & 4 & 4.3334 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Dividing the 2nd row by 1.6667 and reducing the second column. (operating above the diagonal as well as below) gives:

$R_3 - r_3\ R_2$ $\quad$ $R_1 - r_5\ R_2$

$R_4 - r_4\ R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1.5 & -1.2000 & 1.4000 \\ 0 & 1 & 2.9999 & 2.2000 & -2.4000 \\ 0 & 0 & 15.000 & 12.400 & -19.800 \\ 0 & 0 & -5.9998 & -3.4000 & 4.8000 \end{array}\right]$$

Divide the 3rd row by 15.000 and make the elements in the 3rd Column zero.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.04000 & -0.58000 \\ 0 & 1 & 0 & -0.27993 & 1.5599 \\ 0 & 0 & 1 & 0.82667 & -1.3200 \\ 0 & 0 & 0 & 1.5599 & -3.1197 \end{array}\right]$$

Divide the 4th row by 1.5599 and create zero above the diagonal in the fourth column.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -0.49999 \\ 0 & 1 & 0 & 0 & 1.0001 \\ 0 & 0 & 1 & 0 & +0.33326 \\ 0 & 0 & 0 & 1 & -1.9999 \end{array}\right]$$

$x_1 = -0.49999$, $x_2 = 1$, $x_3 = 0,33326$ and $x_4 = -1.99999$

Example Solve using Gauss-Jordan elimination , the following equations system

$$2x_1 - 5x_2 + 5x_3 = 17 \qquad (1)$$

$$-x_1 + 3x_2 = -4 \qquad (2)$$

$$x_1 - 2x_2 + 3x_3 = 9 \qquad (3)$$

Solution

Step 1

$$\begin{bmatrix} 2 & -5 & 5 & 17 \\ -1 & 3 & 0 & -4 \\ 1 & -2 & 3 & 9 \end{bmatrix} \xrightarrow{(R1)\leftrightarrow(R3)}$$

Step 2

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

$R_2 - r_1 R_1$ **where** $r_1 = \frac{a_{21}}{a_{11}} = \frac{-1}{1} = -1$

Step 3

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

$R_3 - r_2 R_1$ **where** $r_2 = \frac{a_{31}}{a_{11}} = \frac{2}{1} = 2$

Step 4

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$R_1 - r_3 R_2$ **where** $r_3 = \frac{a_{12}}{a_{22}} = \frac{-2}{1} = -2$

Step 5

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$R_3 - r_4 R_2$ **where** $r_4 = \frac{a_{32}}{a_{22}} = \frac{-1}{1} = -1$

Step 6 $\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ $\frac{R_3}{2}$

Step 7 $\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $R_2 - r_5\ R_3$ $\textbf{where}\ r_5 = \frac{a_{23}}{a_{33}} = \frac{3}{1} = 3$

Step 8 $\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $R_1 - r_6\ R_3$ $\textbf{where}\ r_6 = 9$

Step 9 $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$x_1 = 1$ $x_2 = -1$, $x_3 = 2$

In iterative methods, initially a solution is assumed and through iterations the actual solution is approached asymptotically.

A) Jacobi iteration B) Gauss-Seidel iteration C) Successive over relaxation

Jacobi iteration Method

The Jacobi method is an iterative method to solve systems of linear algebraic equations. Consider the following system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ . \\ . \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + + a_{nn}x_{nn} = b_n \end{cases}$$

This system can be written under the following form:

$$\begin{cases} x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n\right) \\ . \\ . \\ x_1 = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n,n-1}\right) \end{cases}$$

The general formulation is:

$$x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1,\, j\neq i}^{n} a_{ij}x_j\right] \quad ; i=1,2,\ldots,n$$

Here we start, by an initial guess for x_1; x_2; …; x_n, and we compute the new values for the next iteration. If no good initial guess is available, we can assume each component to be zero.

We generate the solution at the next iteration using the following expression:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1,\, j\neq i}^{n} a_{ij} x_j^k\right]$$; i=1,2,…,n; and for k=,2, …,

convergence The calculation must be stopped if:

$$\left|\frac{x_i^{k+1} - x_i^k}{x_i^k}\right| \leq \varepsilon \qquad \varepsilon$$ is the desired precision.

Example : Use the Jacobi method to approximate the solution of the following system of linear equations.

$$5\,x_1 - 2\,x_2 + 3x_3 = -1$$

$$-\,3x_1 + 9\,x_2 + x_3 = 2$$

$$2\,x_1 - x_2 - 7x_3 = 3$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, write the system in the form

$$x_1 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$

$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$

$$x_3 = -\,\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

Because you do not know the actual solution, choose

$x_1 = 0, \quad x_2 = 0 \quad and \quad x_3 = 0$ Initial approximation

as a convenient initial approximation. So, the first approximation is

$$x_1 = -\frac{1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

$$x_2 = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

$$x_1 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$

$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

Continuing this procedure, you obtain the sequence of approximations shown in Table

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

Example Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \quad\quad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

Convert the set Ax = b in the form of x = Tx + c.

$$\begin{aligned} x_1 &= \frac{1}{10}x_2 - \frac{1}{5}x_3 + \frac{3}{5} \\ x_2 &= \frac{1}{11}x_1 + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11} \\ x_3 &= -\frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_4 - \frac{11}{10} \\ x_4 &= -\frac{3}{8}x_2 + \frac{1}{8}x_3 + \frac{15}{8} \end{aligned}$$

$$\begin{aligned} x_1^{(1)} &= \frac{1}{10}x_2^{(0)} - \frac{1}{5}x_3^{(0)} + \frac{3}{5} \\ x_2^{(1)} &= \frac{1}{11}x_1^{(0)} + \frac{1}{11}x_3^{(0)} - \frac{3}{11}x_4^{(0)} + \frac{25}{11} \\ x_3^{(1)} &= -\frac{1}{5}x_1^{(0)} + \frac{1}{10}x_2^{(0)} + \frac{1}{10}x_4^{(0)} - \frac{11}{10} \\ x_4^{(1)} &= -\frac{3}{8}x_2^{(0)} + \frac{1}{8}x_3^{(0)} + \frac{15}{8} \end{aligned}$$

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0 \text{ and } x_4^{(0)} = 0.$$

$$x_1^{(1)} = \frac{1}{10}(0) - \frac{1}{5}(0) + \frac{3}{5}$$
$$x_2^{(1)} = \frac{1}{11}(0) + \frac{1}{11}(0) - \frac{3}{11}(0) + \frac{25}{11}$$
$$x_3^{(1)} = -\frac{1}{5}(0) + \frac{1}{10}(0) + \frac{1}{10}(0) - \frac{11}{10}$$
$$x_4^{(1)} = -\frac{3}{8}(0) + \frac{1}{8}(0) + \frac{15}{8}$$

$$x_1^{(1)} = 0.6000,$$
$$x_2^{(1)} = 2.2727,$$
$$x_3^{(1)} = -1.1000$$
$$x_4^{(1)} = 1.8750$$

$$x_1^{(2)} = \frac{1}{10}x_2^{(1)} - \frac{1}{5}x_3^{(1)} + \frac{3}{5}$$
$$x_2^{(2)} = \frac{1}{11}x_1^{(1)} + \frac{1}{11}x_3^{(1)} - \frac{3}{11}x_4^{(1)} + \frac{25}{11}$$
$$x_3^{(2)} = -\frac{1}{5}x_1^{(1)} + \frac{1}{10}x_2^{(1)} + \frac{1}{10}x_4^{(1)} - \frac{11}{10}$$
$$x_4^{(2)} = -\frac{3}{8}x_2^{(1)} + \frac{1}{8}x_3^{(1)} + \frac{15}{8}$$

iteration	0	1	2	3
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.0530
$x_3^{(k)}$	0.0000	-1.1000	-0.8052	-1.0493
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309

Gauss-Seidel iteration Method

With the Jacobi method, the values of obtained in the n[th] approximation remain unchanged until the entire (n+1)[th] approximation has been calculated. With the GaussSeidel method, on the other hand, you use the new values of each as soon as they are known. That is, once you have determined from the first equation, its value is then used in the second equation to obtain the new. Similarly, the new and are used in the third equation to obtain the new and so on.

What is the algorithm for the Gauss-Seidel method? Given a general set of *n* equations and *n* unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j}x_j}{a_{11}} \qquad x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}} \qquad x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Hence for any row i

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i 's, one assumes an initial guess for the x_i 's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\in_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

Where x_i^{new} is the recently obtained value of x_i and x_i^{old} is the previous value of x_i

When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example

Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Use $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the initial guess and conduct two iterations.

Solution

The coefficient matrix $[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$

Rewriting the equations, we get $x_1 = \dfrac{1 - 3x_2 + 5x_3}{12}$ $\quad x_2 = \dfrac{28 - x_1 - 3x_3}{5}$ $\quad x_3 = \dfrac{76 - 3x_1 - 7x_2}{13}$

Assuming an initial guess of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Iteration #1

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000 \qquad x_2 = \frac{28 - (0.50000) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

The absolute relative approximate error at the end of the first iteration is

$$|\epsilon_a|_1 = \left|\frac{0.50000 - 1}{0.50000}\right| \times 100 = 100.00\% \qquad |\epsilon_a|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923-1}{3.0923}\right|\times 100 = 67.662\%$$ The maximum absolute relative approximate error is 100.00%

Iteration #2

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76-3(0.14679)-7(3.7153)}{13} = 3.8118$$

At the end of second iteration, the absolute relative approximate error is

$$\left|\in_a\right|_1 = \left|\frac{0.14679-0.50000}{0.14679}\right|\times 100 = 240.61\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153-4.9000}{3.7153}\right|\times 100 = 31.889\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118-3.0923}{3.8118}\right|\times 100 = 18.874\%$$

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.874
3	0.74275	80.236	3.1644	17.408	3.9708	4.0064
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

EXAMPLE Use the Gauss-Seidel method to obtain the solution for

$$\begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{aligned}$$

Solution. First, solve each of the equations for its unknown on the diagonal:

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \qquad (1)$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \qquad (2)$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \qquad (3)$$

let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as the initial guess iterations.

$$x_1 = \frac{7.85 + 0.1(0) + 0.2(0)}{3} = 2.61666667$$

This value, along with the assumed value of x_3 =0, can be substituted into Eq.(2) to calculate

$$x_2 = \frac{-19.3 - 0.1(2.616667) + 0.3(0)}{7} = -2.794524$$

The first iteration is completed by substituting the calculated values for x_1 and x_2 into Eq.(3) to yield

$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794524)}{10} = 7.005610$$

For the second iteration, the same process is repeated to compute

$$x_1 = \frac{7.85 + 0.1(-2.794524) + 0.2(7.005610)}{3} = 2.990557$$

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.005610)}{7} = -2.499625$$

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.499625)}{10} = 7.000291$$